Journal of Statistical Physics, Vol. 124, Nos. 2-4, July/August 2006 (© 2006)
DOI: 10.1007/s10955-006-9064-4

The Development of the Green's Function for the Boltzmann Equation

Shih-Hsien Yu[1]

Received April 18, 2005; accepted October 25, 2005
Published Online: March 21, 2006

We review the particle-like and wave-like property of the Boltzmann equation. This property leads to a sequence of developments on the mathematical theory of the Green's function for the Boltzmann equation.

KEY WORDS: Boltzmann equation, Green's function, initial boundary value problem.

1. INTRODUCTION

The Boltzmannn equation is a fundamental equation for rarefied gases and thermo-non-equilibrium flows. It models the motion of gas flows in terms of a density function $\mathsf{F}(x,t,\xi)$ of particle velocity ξ located at (x,t). The equation for a hard sphere collision model is an integral-differential equation

$$\partial_t \mathsf{F} + \xi \cdot \nabla_x \mathsf{F} = Q(\mathsf{F}),$$

$$Q(\mathsf{F})(\xi) \equiv \int\limits_{\substack{\xi_* \in \mathbb{R}^3,\ \omega \in S^2 \\ (\xi_* - \xi)\cdot\omega > 0}} (-\mathsf{F}(\xi)\mathsf{F}(\xi_*) + \mathsf{F}(\xi')\mathsf{F}(\xi'_*)) \ |(\xi - \xi_*)\cdot\omega| d\omega d\xi_*,$$

$$\begin{cases} \xi' \equiv \xi - [(\xi - \xi_*)\cdot\omega]\omega, \\ \xi'_* \equiv \xi_* + [(\xi - \xi_*)\cdot\omega]\omega. \end{cases}$$

This equation has many industrial applications particularly in the areas related to a condensation-vaporization problem, which is an initial boundary

[1] Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong, SAR, China; e-mail: mashyu@cityu.edu.hk

value problem for the Boltzmann equation. This problem has been extensively studied through numerical computations, asymptotic expansions, and mathematical analysis.

Rigorous mathematical analyses were not mature enough to study any nonlinear problems related to the condensation-vaporization problem until the development of the pointwise estimates of the Green's function $\mathbb{G}(x, t; y, s)$ for an initial boundary value problem.[7] Before the development of the Green's function, there was a classical L^2 energy method for this problem. The energy method is sufficient for a linearized problem but not for a full nonlinear one. This is due to the fact that the energy method requires a smoothness condition in the solution in order to close nonlinearity, but such a required condition generically does not hold at a physical boundary as the numerical solutions indicate.[10] The Green's function approach doesn't require any regularity property of the solution, since the full nonlinear Boltzmann equation is a semi-linear partial differential equation so that one can simply use the pointwise structure of the Green's function and the Picard's iteration to obtain the solution and its pointwise structure.

The study of the Green's function for the Boltzmann equation was first proposed by Cercignani in Ref. 1 for a stationary problem more than twenty years ago. The analysis of the Green's function for an initial boundary value problem requires the basic informations of the Green's function for an initial value problem. To study the initial value problem, one needs to determine carefully the basic physical characteristics of the solution.[5,6] From the consideration related to thermo-equilibrium state, one separates the space-time into two regions: a high Mach number region and a finite Mach number region. From the consideration of solution patterns, one introduces two different decompositions: a long wave-short wave decomposition and a particle-wave decomposition. Those four decompositions are connected by the mixture lemma to result in the Green's function for the initial value problem.

In this paper, we will outline the development of the Green's functions in Ref. 5–7.

2. PRELIMINARIES

The Boltzmann equation,

$$\partial_t \mathsf{F} + \xi \cdot \nabla_x \mathsf{F} = Q(\mathsf{F}) \text{ where } \mathsf{F}(x, t, \xi) \in \mathbb{R}, (x, t, \xi) \in \mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3, \quad (1)$$

consists of two main characteristics: a microscopic particle nature due to $\partial_t + \xi \cdot \nabla_x$ and a macroscopic nature in particle velocity due to the nonlinear binary collision operator Q.

For the hard sphere collision model, the collision operator Q is

$$Q(\mathsf{g})(\xi) \equiv \mathsf{B}(\mathsf{g},\mathsf{g})(\xi),$$

$$\mathsf{B}(\mathsf{g},\mathsf{h})(\xi) \equiv \int\limits_{\substack{\xi_* \in \mathbb{R}^3,\ \omega \in S^2 \\ (\xi_*-\xi)\cdot\omega > 0}} \frac{\left(-\mathsf{g}(\xi)\mathsf{h}(\xi_*) - \mathsf{h}(\xi)\mathsf{g}(\xi_*) + \mathsf{g}(\xi')\mathsf{h}(\xi'_*) + \mathsf{h}(\xi')\mathsf{g}(\xi'_*)\right)}{2} \cdot |(\xi-\xi_*)\cdot\omega| d\omega d\xi_*,$$

$$\begin{cases} \xi' \equiv \xi - [(\xi-\xi_*)\cdot\omega]\omega, \\ \xi'_* \equiv \xi_* + [(\xi-\xi_*)\cdot\omega]\omega. \end{cases} \tag{2}$$

The equilibrium states are defined by $Q^{-1}(0)$:

$$\begin{cases} \mathsf{M} \equiv \mathsf{M}_{[1,0,1]}, \text{ (The absolute Maxwellian state.)} \\ \mathsf{M}_{[\rho,\mathbf{u},\theta]} \equiv \frac{\rho\, e^{-\frac{|\xi-\mathbf{u}|^2}{2\theta}}}{\sqrt{2\pi\theta}}. \text{ (A general Maxwellian state.)} \end{cases}$$

Here the state $(\rho,\mathbf{u},\theta) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^+$ represents bulk density, velocity, and temperature of a thermo-equilibrium state $\mathsf{M}_{[\rho,\mathbf{u},\theta]}$.

Denote the linearized collision operator around an absolute Maxwellian M by L, which is defined by

$$L\mathsf{f} \equiv 2\mathsf{M}^{-1/2}\mathsf{B}(\mathsf{M},\mathsf{M}^{1/2}\mathsf{f});$$

and L can be explicitly written as Ref. 2:

$$\begin{cases} L\mathsf{g}(\xi) = -\nu(\xi)\mathsf{g}(\xi) + \int_{\mathbb{R}^3} k(\xi,\xi_*)\mathsf{g}(\xi_*)d\xi_*, \\ \nu(\xi) \equiv \frac{1}{\sqrt{2\pi}}\left(2e^{-\frac{|\xi|^2}{2}} + 2\left(|\xi| + \frac{1}{|\xi|}\right)\int_0^{|\xi|} e^{-\frac{u^2}{2}}\, du\right), \\ k(\xi,\xi_*) = \frac{2}{\sqrt{2\pi}|\xi-\xi_*|}\exp\left(-\frac{(|\xi|^2-|\xi_*|^2)^2}{8|\xi-\xi_*|^2} - \frac{|\xi-\xi_*|^2}{8}\right) \\ \qquad -\frac{|\xi-\xi_*|}{2}\exp\left(-\frac{(|\xi|^2+|\xi_*|^2)}{4}\right), \\ \nu(\xi) = O(1)(1+|\xi|) > 0. \end{cases} \tag{3}$$

The nonlinear problem Eq. (1) can be written as a perturbation problem as follows

$$\partial_t\mathsf{f} + \xi\cdot\nabla_x\mathsf{f} - L\mathsf{f} = \Gamma(\mathsf{f}) \text{ where } \Gamma(\mathsf{f}) \equiv \mathsf{M}^{-1/2}Q(\mathsf{M}^{-1/2}\mathsf{f}). \tag{4}$$

Remark 1. *The variable* $\mathsf{f} = \mathsf{M}^{-1/2}\mathsf{F}$ *was first introduced by Ref. 4.*

The Green's function $\mathbb{G}(x, t; y, s; \xi, \xi_*)$ for the linear part of Eq. (4) can be defined by $\mathbb{G}(x, t; y, s; \xi, \xi_*) \equiv \mathsf{g}(x - y, t - s, \xi)$, which is the solution of the initial value problem:

$$\begin{cases} \partial_t \mathsf{g} + \xi \cdot \nabla_x \mathsf{g} = L\mathsf{g}, \\ \mathsf{g}(x, 0) = \delta^3(x)\delta^3(\xi - \xi_*). \end{cases} \tag{5}$$

The solution of a general initial value problem

$$\begin{cases} \partial_t \mathsf{h} + \xi \cdot \nabla_x \mathsf{h} = L\mathsf{h}, \\ \mathsf{h}(x, 0, \xi) = \mathsf{h}_0(x, \xi) \end{cases} \tag{6}$$

can be represented:

$$\mathsf{h}(x, t) = \int_{\mathbb{R}^3} \mathbb{G}(x, t; y, 0)\mathsf{h}_0(y)dy = \int_{\mathbb{R}^3} \mathbb{G}(x - y, t)\mathsf{h}_0(y)dy, \tag{7}$$

where $\mathbb{G}(x, t; y, 0)\mathsf{h}_0(y)$ defined an $L^2_\xi(\mathbb{R}^3)$-valued function:

$$\begin{cases} \mathbb{G}(x, t; y, 0)\mathsf{h}_0(y)(\xi) \equiv \int_{\mathbb{R}^3} \mathbb{G}(x, t; y, 0; \xi, \xi_*)\mathsf{h}_0(y, \xi_*)d\xi_*, \\ \mathbb{G}(x, t) \equiv \mathbb{G}(x, t; 0, 0). \end{cases}$$

The solution of Eq. (4) can be represented as

$$\mathsf{f}(x, t) = \int_{\mathbb{R}^3} \mathbb{G}(x - y, t)\mathsf{f}(y, 0)dy + \int_0^t \int_{\mathbb{R}^3} \mathbb{G}(x - y, t - \sigma)\Gamma(\mathsf{f})(y, \sigma)dyds. \tag{8}$$

3. SPECTRUM AND LONG WAVE-SHORT WAVE DECOMPOSITION

It is natural to consider the Fourier transformation of the Green's function. From the Fourier transformations of Eq. (6) and Eq. (7), one has

$$\hat{\mathbb{G}}(\eta, t) = e^{(-i\eta\cdot\xi + L)t}, \tag{9}$$

where

$$\hat{\mathbb{G}}(\eta, t) \equiv \int_{\mathbb{R}^3} e^{-i\eta\cdot x}\mathbb{G}(x, t)dx.$$

One applies inverse Fourier transformation to $\hat{\mathbb{G}}(\eta, t)$, then one can obtain the Green's function $\mathbb{G}(x, t)$ formally as follows

$$\mathbb{G}(x, t) = \int_{\mathbb{R}^3} e^{ix\cdot\eta} e^{(-i\eta\cdot\xi + L)t} d\eta. \tag{10}$$

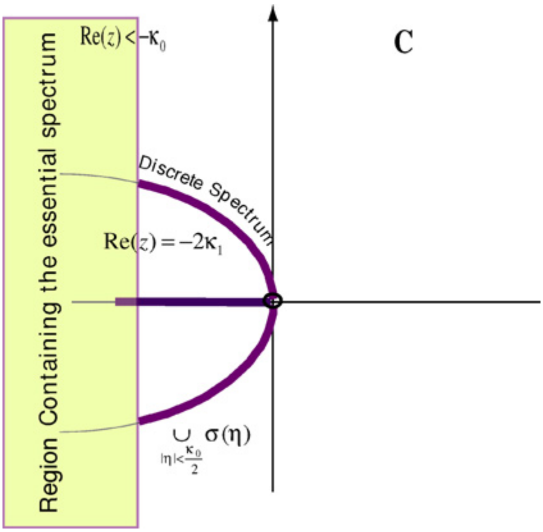

Fig. 1. The diagram of the spectrum.

This leads to the spectrum decomposition of the operator $-i\eta \cdot \xi + L$ for any fixed $\eta \in \mathbb{R}^3$.[3,11] The spectrum $\sigma(\eta)$ of the operator $-i\xi \cdot \eta + L$ is analytic when $|\eta| \ll 1$. The rest of the spectrum is completely contained in the half space $Re(z) < -\kappa_0$ for some $\kappa_0 > 0$. The spectrum $\sigma(\eta)$ is illustrated in Fig. 1 for all $\eta \in \mathbb{R}^3$.

Due to this spectrum property, one introduces the *Long wave-Short wave decomposition* (L-S decomposition)[5]:

$$\begin{cases} \mathbb{G}(x,t) \equiv \mathbb{G}_L(x,t) + \mathbb{G}_S(x,t), \\ \mathbb{G}_L(x,t) \equiv \displaystyle\int_{|\eta| \le \kappa_0/2} e^{ix\cdot\eta} e^{(-i\eta\cdot\xi + L)t} d\eta, \\ \mathbb{G}_S(x,t) \equiv \displaystyle\int_{|\eta| \ge \kappa_0/2} e^{ix\cdot\eta} e^{(-i\eta\cdot\xi + L)t} d\eta; \end{cases} \tag{L-S}$$

and three semi-groups $\mathbb{G}^t$, $\mathbb{G}^t_L$, and $\mathbb{G}^t_S$:

$$\begin{cases} \mathbb{G}^t \mathsf{h}(x) \equiv \displaystyle\int_{\mathbb{R}^3} \mathbb{G}(x-y,t)\mathsf{h}(y)dy, \\ \mathbb{G}^t_L \mathsf{h}(x) \equiv \displaystyle\int_{\mathbb{R}^3} \mathbb{G}_L(x-y,t)\mathsf{h}(y)dy, \\ \mathbb{G}^t_S \mathsf{h}(x) \equiv \displaystyle\int_{\mathbb{R}^3} \mathbb{G}_S(x-y,t)\mathsf{h}(y)dy. \end{cases}$$

From the spectral properties illustrated in Fig. 1, one has that for any given $k \in \mathbb{N}$

$$\begin{cases} \left\|\mathbb{G}_L^t \mathsf{h}\right\|_{H_x^k(L_\xi^2)} \leq O(1)\|\mathsf{h}\|_{L_x^2(L_\xi^2)} \text{ for all } t > 0, \\ \left\|\mathbb{G}_S^t \mathsf{h}\right\|_{L_x^2(L_\xi^2)} \leq O(1)e^{-\frac{\kappa_0 t}{2}}\|\mathsf{h}\|_{L_x^2(L_\xi^2)} \text{ for all } t > 0. \end{cases} \tag{11}$$

This and the Sobolev's inequality yield

$$\left\|\mathbb{G}_L^t \mathsf{h}\right\|_{L_x^\infty(L_\xi^2)} \leq O(1)\|\mathsf{h}\|_{L_x^2(L_\xi^2)}. \tag{12}$$

4. PARTICLE HIERARCHY

Before considering Eq. (5), we consider a model problem for the Green's function[5]:

$$\begin{cases} \partial_t \mathsf{f} + \xi \cdot \nabla_x \mathsf{f} = L\mathsf{f}, \\ \mathsf{f}(x, 0, \xi) = \mathsf{f}_0(x, \xi), \quad \mathsf{f}_0 \equiv 0 \text{ for } |x| \geq 1, \quad \sup_{x,\xi} |\mathsf{f}_0(x, \xi)| \leq 1. \end{cases} \tag{13}$$

Rewrite Eq. (13) as follows.

$$\begin{cases} \partial_t \mathsf{f} + \xi \cdot \nabla_x \mathsf{f} + \nu(\xi)\mathsf{f} = \mathsf{K}\mathsf{f}, \\ \mathsf{f}|_{t=0} = \mathsf{f}_0, \end{cases} \tag{14}$$

where

$$\mathsf{K}\mathsf{f}(x, t, \xi) \equiv \int_{\mathbb{R}^3} k(\xi, \xi_*)\mathsf{f}(x, t, \xi_*)d\xi_*.$$

This is a hyperbolic equation. We denote the solution operator of the following transport equation by $\mathbb{S}^t$

$$\begin{cases} \partial_t \mathsf{h} + \xi \cdot \nabla_x \mathsf{h} + \nu(\xi)\mathsf{h} = 0, \\ \mathsf{h}(x, 0) = \mathsf{h}_0. \end{cases} \tag{15}$$

The operator $\mathbb{S}^t$ can be expressed as follows

$$\begin{cases} \mathbb{S}^t \mathsf{h}_0(x, \xi) = \mathsf{h}_0(x - \xi t)e^{-\nu(\xi)t}, \\ \hat{\mathbb{S}}^t(\eta, t) = e^{-\nu(\xi)t - i\xi\cdot\eta t}. \end{cases} \tag{16}$$

4.1. Essential Kinetic Waves

The integral kernel $k(\xi, \xi_*)$ of the integral operator K contains a singularity at $|\xi - \xi_*| = 0$. Regarding to this singularity, we introduce an essential kinetic equation to approximate the particle behavior of Eq. (14).

$$\begin{cases} \partial_t \mathscr{E} + \xi \cdot \nabla_x \mathscr{E} + \nu \mathscr{E} = \mathsf{K}_0 \mathscr{E}, \\ \mathscr{E}|_{t=0} = \mathsf{f}_0, \end{cases} \tag{17}$$

where

$$\begin{cases} \mathsf{K}_0 \mathscr{E}(x,t,\xi) \equiv \displaystyle\int_{\mathbb{R}^3} k(\xi,\xi_*) \mathscr{E}(x,t,\xi_*) \chi\left(\frac{|\xi-\xi_*|}{\epsilon}\right) d\xi_*, \\ \mathsf{K}_1 \mathscr{E} \equiv \mathsf{K}\mathscr{E} - \mathsf{K}_0 \mathscr{E}, \\ \chi \in C_c^\infty(\mathbb{R}), \ \chi(\tau) = 1 \quad \text{for} \quad |\tau| \le \frac{1}{2}, \ \text{supp}(\chi) \subset [-2,2], \\ 0 < \epsilon \ll 1. \end{cases} \tag{18}$$

The operator K_1 is smooth operator in ξ variable, since $\chi(\frac{|\xi-\xi_*|}{\epsilon})k(\xi,\xi_*)$ is a smooth function.

Remark 2. *The initial data* f_0 *of Eq. (14) is not necessary smooth or continuous.*

The existence of $\mathscr{E}(x,t)$ can be obtained by assuming $\epsilon \ll 1$ in Eq. (18) and by Picard's iterations. The solution $\mathscr{E}(x,t)$ satisfies that

$$\|\mathscr{E}\|_{L_x^2(L_\xi^2)} \le O(1) e^{-\alpha_0 t} \|\mathsf{f}_0\|_{L_x^2(L_\xi^2)} \quad \text{for} \quad \alpha_0 > 0. \tag{19}$$

Set

$$\mathsf{d}_1 \equiv \mathsf{f} - \mathscr{E}.$$

d_1 satisfies

$$\begin{cases} \partial_t \mathsf{d}_1 + \xi \cdot \nabla_x \mathsf{d}_1 + \nu(\xi) \mathsf{d}_1 = \mathsf{K}\mathsf{d}_1 + \mathsf{K}_1 \mathscr{E}, \\ \mathsf{d}_1|_{t=0} \equiv 0. \end{cases} \tag{20}$$

Now, one observes the initial data of d_1 is smooth up to any order, and the source $\mathsf{K}_1 \mathscr{E}$ is smooth in the ξ variable! This regularity property is crucial in the following up development.

4.2. Mixture Lemma

By Picard's iteration, one has

$$\mathsf{d}_1 = (1 + \mathbb{S}^t * \mathsf{K} + \mathbb{S}^t * \mathsf{K}\mathbb{S}^t * \mathsf{K} + \mathbb{S}^t * \mathsf{K}\mathbb{S}^t * \mathsf{K}\mathbb{S}^t * \mathsf{K} + \cdots) \mathbb{S}^t * \mathsf{K}_1 \mathscr{E}. \tag{21}$$

Here, $*$ denotes a convolution operator in t variable.

Remark 3. *This sequence had been used to construct the existence of the operator $e^{-i\xi\cdot\eta+\nu(\xi)t+\mathsf{K}t}$ in Ref. 11.*

Lemma 1. (Mixture Lemma, Ref. 5) *For any given $k \in \mathbb{N}\cup\{0\}$, one has that*

$$\left\|\partial_x^k \mathbb{S}^t * \underbrace{(\mathsf{K}\mathbb{S}^t) * (\mathsf{K}\mathbb{S}^t) * \cdots * (\mathsf{K}\mathbb{S}^t)}_{2k} \mathsf{g}_0\right\|_{L^2_x(L^2_\xi)} = O(1)(\|\mathsf{g}_0\|_{L^2_x(L^2_\xi)} + \|\partial_\xi^k \mathsf{g}_0\|_{L^2_x(L^2_\xi)}), \tag{22}$$

where $O(1)$ depends on k but is independent of g_0.

The proof of this lemma is based on a separation of the time domain into small and large time scales. In the small time scale, one needs to use the characteristic curve method to convert the x-derivative into the ξ-derivative. In the large time scale, one can simply use the Fourier transformation to convert the x-derivative into the ξ-derivative. The combination of the characteristic curve method and the Fourier transformation reveals the particle-like and wave-like duality properties in the Boltzmann equation.

We denote the k-th order mixture operator by

$$\mathbb{M}_k \equiv \mathbb{S}^t * \underbrace{(\mathsf{K}\mathbb{S}^t) * (\mathsf{K}\mathbb{S}^t) * \cdots * (\mathsf{K}\mathbb{S}^t)}_{2k};$$

and express d_1 in terms of $\mathbb{M}_k$:

$$\mathsf{d}_1 = \left(\mathbb{S}^t + \mathbb{S}^t * \mathsf{K}\mathbb{S}^t + (1 + \mathbb{S}^t * \mathsf{K})\sum_{k=1}^{l} \mathbb{M}_k\right) * \mathsf{K}_1\mathscr{E} + \mathbb{W}_l. \tag{23}$$

The equation for $\mathbb{W}_l$ is

$$\begin{cases} \partial_t \mathbb{W}_l + \xi \cdot \nabla_x \mathbb{W}_l - L\mathbb{W}_l = \mathsf{K}\mathbb{M}_l * \mathsf{K}_1\mathscr{E}, \\ \mathbb{W}_l|_{t=0} \equiv 0. \end{cases} \tag{24}$$

By the Mixture Lemma and the smoothness of K_1 in ξ-variable, one has that

$$\begin{aligned} \|\mathsf{K}\mathbb{M}_l * \mathsf{K}_1\mathscr{E}\|_{H^l_x(L^2_\xi)} &= O(1)e^{-\alpha_l t}\left(\|\mathscr{E}\|_{L^2_x(L^2_\xi)} + \|\partial_\xi^l(\mathsf{K}_1\mathscr{E})\|_{L^2_x(L^2_\xi)}\right) \\ &= O(1)e^{-\alpha_l t}\|\mathscr{E}\|_{L^2_x(L^2_\xi)} \end{aligned} \tag{25}$$

for some $\alpha_l > 0$. Then, by Eq. (11), Eq. (19), and Eq. (25) one has that

$$\|\mathbb{W}_l\|_{H_x^l(L_\xi^2)} = O(1)\|\mathsf{f}_0\|_{L_\xi^2(L_\xi^2)}. \tag{26}$$

We introduce a Particle-Wave decomposition (P-W decomposition) for the solution f of Eq. (14):

$$\begin{cases} \mathsf{f} = \underbrace{\mathscr{E} + \mathscr{E}_1}_{\text{particle hierarchy (particle-like component)}} + \underbrace{\mathbb{W}_l}_{\text{(wave-like component)}} \equiv \mathbb{P} + \mathbb{W}_l; \\ \mathscr{E}_1 \equiv \left(\mathbb{S}^t + \mathbb{S}^t * \mathsf{K}\mathbb{S}^t + (1 + \mathbb{S}^t * \mathsf{K}) \sum_{k=1}^{l} \mathbb{M}_l\right) * \mathsf{K}_1\mathscr{E}. \end{cases} \tag{P-W}$$

The particle-like component $\mathbb{P}$ satisfies

$$\begin{cases} \|\mathbb{P}\|_{L_x^\infty(L_\xi^\infty)} \le O(1)e^{-\beta_0 t}\|\mathsf{f}_0\|_{L_x^\infty(L_\xi^\infty)} \quad \text{for} \quad \beta_0 > 0, \\ \|\mathbb{P}\|_{L_x^2(L_\xi^2)} \le O(1)e^{-\beta_0 t}\|\mathsf{f}_0\|_{L_x^2(L_\xi^2)}. \end{cases} \tag{27}$$

This exponential rate of decaying is due to the dissipative nature of $\mathbb{S}^t$.

4.3. The Combination of Long Wave-Short Wave Decomposition and Particle-Wave Decomposition

From

$$(\mathbb{G}_L + \mathbb{G}_S)\mathsf{f}_0 = \mathsf{f} = \mathbb{P} + \mathbb{W}_l, \tag{28}$$

Eq. (26), and Eq. (20) one has

$$\|\mathbb{P} - \mathbb{G}_S\mathsf{f}_0\|_{H_x^l(L_\xi^2)} = \|\mathbb{G}_L\mathsf{f}_0 - \mathbb{W}_l\|_{H_x^l(L_\xi^2)} = O(1)\|\mathsf{f}_0\|_{L_x^2(L_\xi^2)}. \tag{29}$$

From Eq. (27) and Eq. (17), one has that

$$\|\mathbb{P} - \mathbb{G}_S\mathsf{f}_0\|_{L_x^2(L_\xi^2)} = O(1)e^{-\gamma_0 t}\|\mathsf{f}_0\|_{L_x^2(L_\xi^2)} \tag{30}$$

for some $\gamma_0 > 0$, then by Sobolev's inequality one can conclude that

$$\|\mathbb{P} - \mathbb{G}_S\mathsf{f}_0\|_{L_x^\infty(L_\xi^2)} = O(1)e^{-\gamma_0 t}\|\mathsf{f}_0\|_{L_x^2(L_\xi^2)} \quad \text{for some} \quad \gamma_0 > 0. \tag{31}$$

This concludes that the short wave component $\mathbb{G}_S^t \mathsf{f}_0$ is time-asymptotically equivalent to particle-like component $\mathbb{P}$ in the norm $\|\cdot\|_{L_x^\infty(L_\xi^2)}$; and $\mathbb{P}$ can be constructed by the characteristic method and Picard's iteration.

5. LONG WAVE STRUCTURE WITH FINITE MACH NUMBER REGION

We only need to construct $\mathbb{G}_L(x,t)$ in a finite Mach number region

$$|x| \le 2\boldsymbol{c}(t+1),$$

where $\boldsymbol{c}$ is the speed of sound defined by

$$\boldsymbol{c} = \sqrt{\frac{5\theta}{3}}.$$

The condition $|x|/(1+t) < 2\boldsymbol{c}$ is due to the fact that $\hat{\mathbb{G}}_L(\eta,t)$ has compact support. One will not obtain any convergent structure as $|x|/(1+t) \to \infty$ from an inverse Fourier transformation. The inverse Fourier transformation of $\hat{\mathbb{G}}_L(\eta,t)$ is

$$\mathbb{G}_L(x,t) = \int_{|\eta|<\epsilon} e^{ix\cdot\eta+(-i\xi\cdot\eta+L)t}\,d\eta \quad \text{for} \quad |x| \le 2\boldsymbol{c}(t+1). \tag{32}$$

Case 1. Planar Wave
In this case $x,\ \eta \in \mathbb{R}$ and $\xi \in \mathbb{R}^3$, one has

$$\mathbb{G}_L(x,t) = \int_{|\eta|<\epsilon} e^{ix\eta+(-i\xi^1\eta+L)t}\,d\eta \quad \text{for} \quad |x| \le 2\boldsymbol{c}(t+1). \tag{33}$$

When $0 < |\eta| \ll 1$, the operator $-i\xi^1\eta + L$ has three analytic eigenvalues $\sigma_1(\eta)$, $\sigma_2(\eta)$, and $\sigma_3(\eta)$ of the form:

$$\begin{cases} \sigma_1(\eta) = (1+\mathscr{A}_1(\eta^2))i\eta\boldsymbol{c} - (1+\mathscr{B}_1(\eta^2))B_1\eta^2, \\ \sigma_2(\eta) = -(1+\mathscr{B}_2(\eta^2))B_2\eta^2, \\ \sigma_3(\eta) = -(1+\mathscr{A}_1(\eta^2))i\eta\boldsymbol{c} - (1+\mathscr{B}_1(\eta^2))B_1\eta^2, \end{cases} \tag{34}$$

where $\mathscr{A}_j(x)$ and $\mathscr{B}_j(x)$ are all real-valued analytic functions around $x=0$ and vanish at $x=0$. The constants B_j are positive.

Remark 4. *The analyticity of $\sigma_i(\eta)$ is obtained in Ref. 3. The first completed proof of Eq. (34) was given in Ref. 5. With this precise structure, one can carry the inverse Fourier transformation of $\mathbb{G}_L(x,t)$ for 1-D and 3-D.*

One takes the spectral decomposition of the operator $-i\xi^1\eta + L$ in terms of its orthonormal eigenvectors $\psi_i(\eta)$, $(-i\xi^1\eta + L)\psi_j(\eta) = \sigma_j(\eta)\psi_j(\eta)$ as follows

$$e^{(-i\xi^1\eta+L)t} = \sum_{j=1}^{3} e^{\sigma_j(\eta)t}\psi_j(\eta) \otimes \langle\psi_j(\eta)| + e^{(-i\xi^1\eta+L)t}\Pi_\eta^\perp, \tag{35}$$

where $\Pi_\eta^\perp \mathsf{h} \equiv \mathsf{h} - \sum_{j=1}^{3}(\psi_j(\eta), \mathsf{h})\psi_j(\eta)$. Due to the spectrum gap in $\sigma(\eta)$, one has that

$$|e^{(-i\xi^1\eta+L)t}\Pi_\eta^\perp| \le O(1)e^{-t/C} \quad \text{for some} \quad C > 0.$$

One can expand the other component as follows.

$$\int_{|\eta|\le\epsilon} \sum_{j=1}^{3} e^{ix\eta+\sigma_j(\eta)t}\psi_j(\eta) \otimes \langle\psi_j(\eta)|\, d\eta$$

$$= \int_{|\eta|\le\epsilon} \sum_{j=1}^{3} e^{ix\eta+(i\sigma_j(0)\eta t+\frac{\sigma_j''(0)}{2}\eta^2 t)+Error_j(\eta)t}\psi_j(\eta) \otimes \langle\psi_j(\eta)|\, d\eta,$$

where $Error_j(\eta)$ is an analytic function in η of order η^3.

Due to the analyticities of $Error_j(\eta)$ and $\psi_j(\eta)$ around $\eta = 0$, the asymptotics $Error_j(\eta) = O(1)\eta^3$ for $|\eta| \ll 1$, and $|x|/(t+1) < 2\boldsymbol{c}$, one can apply the complex contour integral method to obtain that[5]:

$$\|\mathbb{G}_L(x,t)\|_{L^2_\xi} \le O(1)\left[\frac{e^{-\frac{|x+\boldsymbol{c}t|^2}{C(t+1)}}}{\sqrt{t+1}} + \frac{e^{-\frac{|x-\boldsymbol{c}t|^2}{C(t+1)}}}{\sqrt{t+1}} + \frac{e^{-\frac{x^2}{C(t+1)}}}{\sqrt{t+1}}\right]$$

$$\text{for } |x| \le 2\boldsymbol{c}(1+t) \quad \text{for some} \quad C > 0. \tag{36}$$

Remark 5. *To use the complex contour integral to obtain an exponentially sharp estimate of Green's function was first done by Ref. 9 for the compressible Navier-Stokes equation in 1-D.*

Case 2. Waves in 3-D

The difficulty in this case primarily is due to the fact that the eigenfunctions $\psi_j(\eta)$ and spectrum $\sigma_j(\eta)$ are no more analytic functions in $\eta \in \mathbb{R}^3$ around $\eta = 0$. One needs to use $SO(3)$ symmetry to resolve the difficulty.

The reduction procedure is given as follows. For any given non-zero $\eta \in \mathbb{R}^3$, one can have a $\mathfrak{g} \in SO(3)$ so that $\mathfrak{g} : \frac{\eta}{|\eta|} \longmapsto (1, 0, 0)$. This element $\mathfrak{g}$ induces an $SO(3)$ group action on $\mathbb{R}^3 \times \mathbb{R}^3$:

$$\begin{cases} \mathfrak{g} : (x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3 \longmapsto (\mathfrak{g}x, \mathfrak{g}\xi) \in \mathbb{R}^3 \times \mathbb{R}^3, \\ \mathfrak{g} : \mathsf{h} \in L^2_\xi \longmapsto \mathfrak{g}\mathsf{h} \in L^2_\xi \text{ defined by } \mathfrak{g}\mathsf{h}(\xi) \equiv \mathsf{h}(\mathfrak{g}\xi). \end{cases}$$

Under this transformation, one can show that the linear collision operator L is invariant under $\mathfrak{g}$:

$$\mathfrak{g}L\mathfrak{g}^{-1} = L,$$

and

$$\mathfrak{g}(-i\eta \cdot \xi + L)\mathfrak{g}^{-1} = -i|\eta|\xi^1 + L.$$

This yields that $(-i\eta \cdot \xi + L)$ and $-i|\eta|\xi^1 + L$ are conjugate. This concludes that $\sigma(\eta) = \sigma(|\eta|)$ and reduces the spectral decomposition of the operator $(-i\eta \cdot \xi + L)$ for $\eta \in \mathbb{R}^3$ into that of the operator $-i|\eta|\xi^1 + L$, which already has been obtained for 1-D case in Eq. (35) as follows.

$$\begin{aligned} e^{(-i\xi\cdot\eta+L)t} &= \mathfrak{g}^{-1} e^{(-i\xi^1|\eta|+L)t}\mathfrak{g} \\ &= \mathfrak{g}^{-1}\left(\sum_{j=1}^{5} e^{\sigma_j(|\eta|)t}\psi_j(|\eta|) \otimes \langle \psi_j(|\eta|)| + e^{(-i\xi^1|\eta|+L)t}\Pi^{\perp}_{|\eta|}\right)\mathfrak{g}. \end{aligned} \tag{37}$$

Furthermore, one has the spectrum property for $|\eta| \ll 1$

$$\begin{cases} \sigma_1(\eta) = (1 + \mathscr{A}_1(|\eta|^2))i|\eta|\boldsymbol{c} - (1 + \mathscr{B}_1(|\eta|^2))B_1|\eta|^2, \\ \sigma_2(\eta) = -(1 + \mathscr{B}_2(|\eta|^2))B_2|\eta|^2, \\ \sigma_3(\eta) = -(1 + \mathscr{A}_1(|\eta|^2))i|\eta|\boldsymbol{c} - (1 + \mathscr{B}_1(|\eta|^2))B_1|\eta|^2, \\ \sigma_4(\eta) = -(1 + \mathscr{B}_4(|\eta|^2))B_4|\eta|^2, \\ \sigma_5(\eta) = \sigma_4(\eta). \end{cases} \tag{38}$$

Here, functions $\mathscr{A}_j(x)$ and $\mathscr{B}_j(x)$ are all real analytic functions in $x \in \mathbb{R}$.[6] The functions $\sigma_1(\eta)$ and $\sigma_3(\eta)$ are not analytic functions in $\eta \in \mathbb{R}^3$ due to the factor $|\eta|$. We need special symmetries for the spectral decomposition of $e^{(-i\xi\cdot\eta+L)t}$ to obtain the analyticity in $\eta \in \mathbb{R}^3$. The symmetries are

Huygen Pairing	$\hat{\mathfrak{H}}(\eta, t) \equiv \mathfrak{g}^{-1} \sum_{j\in\{1,3\}} e^{\sigma_j(\lvert\eta\rvert)t} \psi_j(\lvert\eta\rvert) \otimes \langle \psi_j(\lvert\eta\rvert) \rvert\, \mathfrak{g}$ $-\mathfrak{g}^{-1} \sum_{j\in\{1,3\}} e^{\sigma_j(\lvert\eta\rvert)t} \mathsf{P}_0^m \psi_j(\lvert\eta\rvert) \otimes \langle \mathsf{P}_0^m \psi_j(\lvert\eta\rvert) \rvert\, \mathfrak{g}$
Contact Pairing	$\hat{\mathfrak{C}}(\eta, t) \equiv \mathfrak{g}^{-1} \left(e^{\sigma_2(\lvert\eta\rvert)t} \psi_2(\lvert\eta\rvert) \otimes \langle \psi_2(\lvert\eta\rvert) \rvert \right) \mathfrak{g}$
Rotational Pairing	$\hat{\mathfrak{R}}_n(\eta, t) \equiv e^{\sigma_4(\lvert\eta\rvert)t} \mathfrak{g}^{-1} \left(\Pi_{\lvert\eta\rvert} - \sum_{j=1}^{3} \psi_j(\lvert\eta\rvert) \otimes \langle \psi_j(\lvert\eta\rvert) \rvert \right) \mathfrak{g}$ $+\mathfrak{g}^{-1} \sum_{j\in\{1,3\}} \mathsf{P}_0^m \psi_j(\lvert\eta\rvert) \otimes \langle \mathsf{P}_0^m \psi_j(\lvert\eta\rvert) \rvert\, \mathfrak{g}$
Riesz Pairings	$\hat{\mathfrak{P}\mathfrak{R}}(\eta, t) \equiv \sum_{j\in\{1,3\}} (e^{\sigma_j(\lvert\eta\rvert)t} - e^{\sigma_4(\lvert\eta\rvert)t})$ $\cdot \mathfrak{g}^{-1} \mathsf{P}_0^m \psi_j(\lvert\eta\rvert) \otimes \langle \mathsf{P}_0^m \psi_j(\lvert\eta\rvert) \rvert\, \mathfrak{g},$ $\hat{\mathfrak{P}\mathfrak{R}}(\eta, t) = \hat{\mathfrak{P}\mathfrak{R}}_1(\eta, t) + \hat{\mathfrak{P}\mathfrak{R}}_2(\eta, t),$ $\hat{\mathfrak{P}\mathfrak{R}}_1(\eta, t) \equiv \sum_{j\in\{1,3\}} e^{\sigma_j(\lvert\eta\rvert)t} \mathfrak{g}^{-1} \mathsf{P}_0^m \psi_j(\lvert\eta\rvert) \otimes \langle \mathsf{P}_0^m \psi_j(\lvert\eta\rvert) \rvert\, \mathfrak{g}$ $-e^{\mathsf{A}_1^2(\lvert\eta\rvert^2)t} \sum_{j\in\{1,3\}} \mathfrak{g}^{-1} \mathsf{P}_0^m \psi_j(\lvert\eta\rvert) \otimes \langle \mathsf{P}_0^m \psi_j(\lvert\eta\rvert) \rvert\, \mathfrak{g},$ $\hat{\mathfrak{P}\mathfrak{R}}_2(\eta, t) \equiv (e^{B_2(\lvert\eta\rvert^2)t} - e^{\sigma_4(\lvert\eta\rvert)t})$ $\cdot \sum_{j\in\{1,3\}} \mathfrak{g}^{-1} \mathsf{P}_0^m \psi_j(\lvert\eta\rvert) \otimes \langle \mathsf{P}_0^m \psi_j(\lvert\eta\rvert) \rvert\, \mathfrak{g},$

(39)

where

$$\mathsf{P}_0^m \mathsf{h} \equiv \sum_{j=1}^{3} \mathsf{M}^{1/2} \xi^j (\mathsf{M}^{1/2} \xi^j, \mathsf{h}).$$

In the above, the transformation $\mathfrak{g}$ is also a function ofη. Under the re-arrangements by the above symmetries, each pairing are analytic in $\eta \in \mathbb{R}^3$ around $\eta = 0$, though some terms in the pairings are not analytic in $\eta \in \mathbb{R}^3$. (The proof is very lengthy.)

Here, the Riesz pairing contains a factor $\frac{\eta^j \eta^l}{|\eta|^2}$, which is the Riesz's transformation in terms of Fourier variable. This symmetry arouses due to the higher space dimension.

One can apply the complex contour integral to evaluate the inverse Fourier transformation of the above pairings. Finally, one can conclude that there exists

$C > 0$ such that for any $|x| \le 2\boldsymbol{c}(t+1)$

$$\|\mathfrak{H}(x,t)\|_{L^2_\xi} = \left\| \iiint_{|\eta|\le\epsilon} e^{ix\cdot\eta}\hat{\mathfrak{H}}(\eta,t)d\eta \right\|_{L^2_\xi} \le C\left[\frac{e^{-\frac{(|x|-\boldsymbol{c}t)^2}{Ct}}}{(1+t)^2} + e^{-t/C}\right], \quad (40a)$$

$$\|\mathfrak{C}(x,t)\|_{L^2_\xi} = \left\| \iiint_{|\eta|\le\epsilon} e^{ix\cdot\eta}\hat{\mathfrak{C}}(\eta,t)d\eta \right\|_{L^2_\xi} \le C\left[\frac{e^{-\frac{|x|^2}{Ct}}}{(1+t)^{3/2}} + e^{-t/C}\right], \quad (40b)$$

$$\|\mathfrak{R}_n(x,t)\|_{L^2_\xi} = \left\| \iiint_{|\eta|\le\epsilon} e^{ix\cdot\eta}\hat{\mathfrak{R}}_n(\eta,t)d\eta \right\|_{L^2_\xi} \le C\left[\frac{e^{-\frac{|x|^2}{Ct}}}{(1+t)^{3/2}} + e^{-t/C}\right], \quad (40c)$$

$$\|\mathfrak{P}\mathfrak{R}_2(x,t)\|_{L^2_\xi} = \left\| \iiint_{|\eta|\le\epsilon} e^{ix\cdot\eta}\hat{\mathfrak{P}\mathfrak{R}}_2(\eta,t)d\eta \right\|_{L^2_\xi} \le C\left[\frac{e^{-\frac{|x|^2}{C(t+1)}}}{(1+t)^{\frac{3}{2}}} + e^{-t/C}\right], \quad (40d)$$

$$\|\mathfrak{P}\mathfrak{R}_1(x,t)\|_{L^2_\xi} \le C \begin{cases} \dfrac{e^{-\frac{(|x|-\boldsymbol{c}t)^2}{C(t+1)}}}{(1+t)^2} + e^{-t/C} & \text{for} \quad |x| \in [\boldsymbol{c}t, 2\boldsymbol{c}t], \\ \dfrac{1}{t(|x|+\sqrt{t+1})} & \text{for} \quad |x| \le \boldsymbol{c}t. \end{cases} \quad (40e)$$

From Eq. (40), one has that for $|x| \le 2\boldsymbol{c}(t+1)$ there exists $C > 0$ such that

$$\|\mathbb{G}_L(x,t)\|_{L^2_\xi} \le O(1)\left(\frac{e^{-\frac{(|x|-\boldsymbol{c}t)^2}{C(t+1)}}}{(1+t)^2} + \frac{e^{-\frac{|x|^2}{C(t+1)}}}{(1+t)^{/3/2}} + e^{-t/C}\right) + C\begin{cases} 0 & \text{for} \quad |x| \in [\boldsymbol{c}t, 2\boldsymbol{c}t], \\ \dfrac{1}{t(|x|+\sqrt{t+1})} & \text{for} \quad |x| \le \boldsymbol{c}t. \end{cases} \quad (40f)$$

From Eq. (27), Eq. (31), and Eq. (40f) one has that[6]:

$$\|\mathbb{G}^t \mathsf{f}_0(x,t)\|_{L^2_\xi} \le O(1)\left(\frac{e^{-\frac{(|x|-\boldsymbol{c}t)^2}{C(t+1)}}}{(1+t)^2} + \frac{e^{-\frac{|x|^2}{C(t+1)}}}{(1+t)^{/3/2}} + e^{-t/C}\right) + C\begin{cases} 0 & \text{for} \quad |x| \in [\boldsymbol{c}t, 2\boldsymbol{c}t], \\ \dfrac{1}{t(|x|+\sqrt{t+1})} & \text{for} \quad |x| \le \boldsymbol{c}t. \end{cases} \quad (40g)$$

6. WAVES OUTSIDE FINITE MACH REGION

In the region $|x| \geq 2\boldsymbol{c}(t+1)$, the structure of Boltzmann equation is not necessary an analogy to continuum fluid mechanics. In this region, one just needs to use the following energy method to ensure the exponential decaying structure of $\mathbb{W}_l(x,t)$ which is given in Eq. (P-W).

From Eq. (24), one has that

$$\int_{\mathbb{R}^3} e^{\delta(x^1-\frac{3}{2}ct)}\left(\mathbb{W}_l, \partial_t\mathbb{W}_l + \xi\cdot\nabla_x\mathbb{W}_l - L\mathbb{W}_l - \mathsf{K}\mathbb{M}_l * \mathsf{K}_1\mathscr{E}\right)dx = 0 \quad \text{for some} \quad \delta > 0. \tag{41}$$

This gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} e^{\delta(x^1-\frac{3}{2}ct)}(\mathbb{W}_l,\mathbb{W}_l)dx\\&+\int_{\mathbb{R}^3} e^{\delta(x^1-\frac{3}{2}ct)}\left[\left(\mathbb{W}_l, \delta\left(-\xi^1+\frac{3}{2}\boldsymbol{c}\right)\mathbb{W}_l\right)-(\mathbb{W}_l, L\mathbb{W}_l)\right]dx\\&-\int_{\mathbb{R}^3} e^{\delta(x^1-\frac{3}{2}ct)}(\mathbb{W}_l, \mathsf{K}\mathbb{M}_l * \mathsf{K}_1\mathscr{E})dx = 0.\end{aligned} \tag{42}$$

There exist $\delta > 0$ and $C > 0$ such that

$$0 < \frac{\delta}{C}(\mathbb{W}_l,\mathbb{W}_l) \leq \delta\left(\mathbb{W}_l, \left(-\xi^1+\frac{3}{2}\boldsymbol{c}\right)\mathbb{W}_l\right) - (\mathbb{W}_l, L\mathbb{W}_l). \tag{43}$$

This estimate was first discovered by Ref. 12.

From Eq. (43) and Eq. (42), one can conclude that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} e^{\delta(x^1-\frac{3}{2}ct)}(\mathbb{W}_l,\mathbb{W}_l)dx + \frac{\delta}{2C}\int_{\mathbb{R}^3} e^{\delta(x^1-\frac{3}{2}ct)}(\mathbb{W}_l,\mathbb{W}_l)dx\\&\leq \frac{2C}{\delta}\int_{\mathbb{R}^3} e^{\delta(x^1-\frac{3}{2}ct)}\|\mathsf{K}\mathbb{M}_l * \mathsf{K}_1\mathscr{E}\|^2_{L^2_\xi}dx \leq O(1)e^{-\alpha_0 t} \quad \text{for some} \quad \alpha_0 > 0.\end{aligned} \tag{44}$$

This yields that

$$\int_{\mathbb{R}^3} e^{\delta(x^1-\frac{3}{2}ct)}(\mathbb{W}_l,\mathbb{W}_l)dx \leq O(1)e^{-t/C_0} \quad \text{for some} \quad C_0 > 0. \tag{45}$$

With this lower energy estimate, with uniformly high order estimate Eq. (26), and with the fact that the weighted function $e^{\delta(x^1-ct)}$ can be chosen in the form $e^{\delta(x\cdot\boldsymbol{n}-ct)}$ for any $|\boldsymbol{n}| = 1$, there exists $C > 0$ such that

$$\|\mathbb{W}_l(x,t)\|_{L^2_\xi} \leq O(1)e^{-(|x|+t)/C} \quad \text{for} \quad |x| \geq 2\boldsymbol{c}(t+1). \tag{46}$$

It results in

$$\|\mathbb{G}^t \mathsf{f}_0(x,t)\|_{L^2_\xi} \le O(1)e^{-(|x|+t)/C} \quad \text{for} \quad |x| \ge 2\boldsymbol{c}(t+1). \tag{47}$$

Finally, there exists $C > 0$ satisfying the following[6]:

$$\|\mathbb{G}^t \mathsf{f}_0(x,t)\|_{L^2_\xi} \le O(1)\left(\frac{e^{-\frac{(|x|-\boldsymbol{c}t)^2}{C(t+1)}}}{(1+t)^2} + \frac{e^{-\frac{|x|^2}{C(t+1)}}}{(1+t)^{3/2}} + e^{-(t+|x|)/C}\right)$$
$$+C\begin{cases} 0 \text{ for } |x| \ge \boldsymbol{c}t, \\ \dfrac{1}{t(|x|+\sqrt{t+1})} \quad \text{for} \quad |x| \le \boldsymbol{c}t. \end{cases} \tag{48}$$

6.1. The Delta Functions

One rewrites Eq. (5) as follows

$$\begin{cases} \partial_t \mathsf{g} + \xi \cdot \mathsf{g} + \nu \mathsf{g} = \mathsf{K}\mathsf{g}, \\ \mathsf{g}(x,0) = \mathsf{g}_0(x) \equiv \delta^3(x)\delta^3(\xi - \xi_*). \end{cases}$$

Then, expand g as a finite sum of the Picard's iteration:

$$\mathsf{g}(x,t) = \sum_{n=0}^{l-1} \mathsf{J}_n(x,t) + \mathsf{R}_l(x,t), \tag{49}$$

where

$$\mathsf{J}_n(x,t) \equiv \mathbb{S}^t * \underbrace{(\mathsf{K}\mathbb{S}^t) * \cdots * (\mathsf{K}\mathbb{S}^t)}_{n} \mathsf{g}_0(x).$$

It is clear from the characteristic carve method that

$$\begin{cases} \mathsf{J}_0(x,t,\xi) = e^{-\nu(\xi_*)t}\delta^3(x-\xi t)\delta^3(\xi-\xi_*) \text{ (delta function in } x \text{ and } \xi), \\ \mathsf{J}_1(x,t,\xi) = \displaystyle\int_0^t K(\xi,\xi_0)e^{-\nu(\xi)(t-s)-\nu(\xi_0)s}\delta(x-(t-s)\xi - s\xi_0)ds, \\ \mathsf{J}_2(x,t,\xi) = \displaystyle\int_0^t \int_{\mathbb{R}^3}\int_0^{s_1} e^{-\nu(\xi)(t-s_1)-\nu(\xi_1)(s_1-s)-\nu(\xi_0)s}K(\xi,\xi_1) \\ \qquad \cdot K(\xi_1,\xi_0)\delta(x-(t-s_1)\xi-(s_1-s)\xi_1 - s\xi_0)ds\,d\xi_1\,ds_1, \\ \|\mathsf{J}_n(x,t)\|_{L^\infty_\xi} \le O(1)e^{-(|x|+t)/C_n} \quad \text{for} \quad x\in\mathbb{R}^3, t>0, \\ \qquad n \ge 2 \text{ for some constant } C_n > 0. \end{cases} \tag{50}$$

From the above properties, when $n \geq 3$, the functions $\mathsf{J}_n(x, t, \xi)$ are in the space $L^\infty_x(L^\infty_\xi)$ with exponentially decaying structure in x and t, and the remainder term R_l satisfies

$$\partial_t \mathsf{R}_l + \xi \cdot \nabla_x \mathsf{R}_l - L\mathsf{R}_l = \underbrace{(\mathsf{K}\mathbb{S}^t) * \cdots * (\mathsf{K}\mathbb{S}^t)}_{l}\, \mathsf{g}_0.$$

Thus, the functional theory in Eq. (48) can be applied to $\mathsf{R}_l(x, t) = \int_0^t \int_{\mathbb{R}^3} \mathbb{G}(x - y, t - s)\mathsf{K}\mathsf{J}_l(y, s)dyds$ to result in that for some $C > 0$,[(6)]:

$$\|\mathsf{R}_l(x, t)\|_{L^2_\xi} \leq O(1)\left(\frac{e^{-\frac{(|x|-ct)^2}{C(t+1)}}}{(1+t)^2} + \frac{e^{-\frac{|x|^2}{C(t+1)}}}{(1+t)^{3/2}} + e^{-(t+|x|)/C}\right)$$
$$+C\begin{cases} 0 \text{ for } |x| \geq \boldsymbol{c}t, \\ \dfrac{1}{t(|x| + \sqrt{t+1})} \quad \text{for} \quad |x| \leq \boldsymbol{c}t. \end{cases} \tag{51}$$

From Eq. (50) and Eq. (51), one can obtain the pointwise behavior of the Green's function $\mathbb{G}(x, t, \xi, \xi_*)$ instead of the semi-group type functional property.

7. ON THE INITIAL BOUNDARY VALUE PROBLEMS

The solution of the initial-boundary value problem

$$\begin{cases} \partial_t \mathsf{h} + \xi^1 \partial_x \mathsf{h} = L\mathsf{h} \quad \text{for} \quad x, t > 0, \\ \mathsf{h}(x, 0) = \mathsf{h}_0(x), \\ \mathsf{h}(0, t)|_{\xi^1>0} \equiv 0, \end{cases} \tag{52}$$

can be represented as follows

$$\mathsf{h}(x, t) = \int_0^\infty \mathbb{G}(x - y, t)\mathsf{h}_0(y)dy + \int_0^t \mathbb{G}(x, t - \sigma)[\xi^1 \mathsf{h}(0, s)]ds. \tag{53}$$

The only unknown variable in this representation is the boundary data $\mathsf{h}(0, \sigma)|_{\xi^1<0}$. One can have various approaches to construct the boundary data $\mathsf{h}(0, t)|_{\xi^1<0}$ accurately.

We illustrate a simple example of the Green's function approach for an initial boundary value problem.

Suppose that the Mach number $\mathscr{M}$ for the Maxwellian state in L satisfies

$$\mathscr{M} < -1, \tag{54}$$

and $y = 0$ defined in the intial data h_0 of Eq. (61).

From Eq. (52) one has that (similar to Eq. (41)) for any $\delta \in \mathbb{R}$

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty e^{\delta x}(\mathsf{h},\mathsf{h})dx - \delta\int_0^\infty e^{\delta x}(\mathsf{h},\xi^1\mathsf{h})dx - (\mathsf{h},\xi^1\mathsf{h})|_{x=0} - \int_0^\infty e^{\delta x}(\mathsf{h},L\mathsf{h})dx = 0. \tag{55}$$

Due to the imposed boundary condition $\mathsf{h}(0,t)|_{\xi^1>0} = 0$, the term

$$(\mathsf{h},\xi^1\mathsf{h})|_{x=0} > 0. \tag{56}$$

From the condition $\mathscr{M} < -1$, there exists $\delta > 0$ such that

$$\Lambda\frac{\delta}{2}(\mathsf{h},\mathsf{h}) \le -\delta(\mathsf{h},\xi^1\mathsf{h}) - (\mathsf{h},L\mathsf{h}) \text{ for some } \Lambda > 0. \tag{57}$$

From Eq. (55), Eq. (56), and Eq. (57), one has the estimates that

$$e^{\Lambda\delta t}\int_0^\infty e^{\delta y}(\mathsf{h},\mathsf{h})dy - \int_0^t e^{\Lambda\delta s}(\mathsf{h},\xi^1\mathsf{h})|_{x=0}ds \le \int_0^\infty e^{\delta y}(\mathsf{h}_0,\mathsf{h}_0)dy. \tag{58}$$

This gives the global energy estimate of the boundary data. Next, one uses the decomposition in Eq. (49) to decompose the Green's function $\mathbb{G}(x,t)$:

$$\mathbb{G}(x,t) = \mathsf{J}_0(x,t) + \mathsf{J}_1(x,t) + \mathsf{J}_2(x,t) + \mathsf{R}_2(x,t).$$

Substitute this decomposition into the representation Eq. (53), then it yields that

$$\mathsf{h}(x,t) = \int_0^t \left(\sum_{n=0}^{2}\mathsf{J}_n(x,t-s) + \mathsf{R}_2(x,t-s)\right)\xi^1\mathsf{h}(0,s)ds + \int_0^\infty \mathbb{G}(x,t)\mathsf{h}_0(y)dy. \tag{59}$$

Then, from Eq. (58), Eq. (50), and

$$\|\mathsf{R}_2(x,t)\|_{L^2_\xi} \le O(1)\left(\frac{e^{-\frac{(x-c(\mathscr{M}\pm1)t)^2}{C(t+1)}}}{\sqrt{(t+1)}} + \frac{e^{-\frac{(x-c\mathscr{M}t)^2}{C(t+1)}}}{\sqrt{(t+1)}} + e^{-(|x|+t)/C}\right)$$

for some $C > 0$, one has that

$$\|\mathsf{h}(x,t)\|_{L^2_\xi} \le O(1)e^{-\delta(\Lambda t+x)/C} \quad \text{for all } x,t>0 \quad \text{for some constant } C > 0. \tag{60}$$

For a general Mach number $\{|\mathscr{M}| < 1\} \cap \{|\mathscr{M}| \notin \{0,1\}\}$, the estimate Eq. (57) can not hold for any choice $\Lambda > 0$. The energy estimate of the boundary data $\mathsf{h}(0,t)$ in Eq. (58) is not valid. A procedure called upwind damping is introduced to construct an accurate approximation to the boundary data.[7] Finally, one has that

Theorem 1. *Let y be a given positive constant. Suppose that the initial data* h_0 *in Eq. (52) satisfies*

$$\begin{cases} \mathsf{h}_0(x) \equiv 0 \, for \, |x-y| > 1 \\ \|\mathsf{h}_0\|_{L^\infty_x(L^\infty_{\xi,\beta})} \le 1, \quad \beta \ge 3/2, \\ \|\mathsf{h}_0(x)\|_{L^\infty_{\xi,\beta}} \equiv \sup_{\xi\in\mathbb{R}^3}(1+|\xi|)^\beta |\mathsf{h}_0(x,\xi)|, \end{cases} \tag{61}$$

and $\mathscr{M}$ *Mach number of the Maxwellian in the linearized collision L satisfying* $-1 < \mathscr{M} < 0$ *. Then, there exists* $C > 0$ *such that*

$$\|\mathsf{h}(x,t)\|_{L^2_\xi} \le O(1)\sum_{j=1}^{3}\left[\frac{e^{-\frac{|x-y_j-(\mathscr{M}-c)t|^2}{C(t+1)}}}{\sqrt{t+1}} + \frac{e^{-\frac{|x-y_j-(\mathscr{M}+c)t|^2}{C(t+1)}}}{\sqrt{t+1}} + \frac{e^{-\frac{(x-y_j-\mathscr{M}t)^2}{C(t+1)}}}{\sqrt{t+1}}\right]$$
$$+ O(1)e^{-(|x-y|+t)/C}, \tag{62}$$

where $\{y_1, y_2, y_3\} = \{y, \frac{(\mathscr{M}+1)}{\mathscr{M}-1}y, \frac{\mathscr{M}}{\mathscr{M}-1}y\}$.

With this estimate, the first global existence theorem of a nonlinear half space problem with general Mach number follows in Ref. 7.

ACKNOWLEDGMENTS

The work described in this paper was fully supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China CityU 103304.

REFERENCES

1. C. Cercignani, *The Boltzmann Equation and Its Applications* (Springer Verlag, Berlin, 1988).
2. C. Cercignani, R. Illner, and M. Pulvirenti, The mathematical theory of dilute gases, *Applied Mathematical Sciences* Vol. **106** (Springer-Verlag, New York, 1994).
3. R. Ellis and R. M. Pinsky, The first and second fluid approximations to the linearized Boltzmann equation, *J. Math. Pures Appl.* **54**(9):125–156 (1975).
4. D. Hilbert, *Grundzüge einer Allgemeinen Theorie der Linearen Integralgleichungen* (Teubner, Leipzig, 1912).
5. T.-P. Liu and S.-H. Yu, The Green's function and large-time behavior of solutions for the one-dimensional Boltzmann equation, *Comm. Pure Appl. Math.* **57**(12):1543–1608 (2004).
6. T.-P. Liu and S.-H. Yu, Green's Function and Large-Time behavior of Solutions of Boltzmann Equation, 3-D Waves, to appear.
7. T.-P. Liu and S.-H. Yu, Initial-boundary value problem for boltzmann equation, planar waves, to appear
8. T.-P. Liu and W. Wang, The pointwise estimates of diffusion wave for the Navier–Stokes systems in odd multi-dimensions, *Comm. Math. Phys.* **196**(1):145–173 (1998).
9. T.-P. Liu and Y. Zeng, Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws, *Mem. Amer. Math. Soc.* **125**(599) (1997).

10. Y. Sone, *Kinetic Theory and Fluid Dynamics* (Birkhauser, 2002).
11. S. Ukai, Les solutions globales de l'équation de Boltzmann dans l'espace tout entier et dans le demi-espace, *C. R. Acad. Sci. Paris Ser. A-B* **282**(6):Ai, A317–A320 (1976).
12. S. Ukai, T. Yang, and S.-H. Yu, Nonlinear boundary layers of the Boltzmann equation. I. Existence. *Comm. Math. Phys.* **236**(3):373–393 (2003).